

Solutions to *Mathematical Methods of Classical Mechanics*

August 21, 2026

Chapter 1

PROBLEM p. 12: Determine with what velocity a stone must be thrown in order that it fly infinitely far from the surface of the Earth.

SOLUTION: Assume that $x(t) : \mathbb{R} \rightarrow \mathbb{R}$ is the path of a thrown stone. Thus we have that

$$\ddot{x} = -\frac{gr_0^2}{(r_0 + x)^2} \quad (1.1)$$

where $g = 9.8 \frac{\text{m}}{\text{s}^2}$ and r_0 is the radius of the Earth, approximately 6,371 km. We also assume the stone begins at the surface of the Earth, so that $x(0) = 0$ and that $x(t)$ increases unboundedly, which is to say that the stone escapes Earth's gravitation.

The chain rule gives us the following

$$\frac{d}{dt} \frac{\dot{x}^2}{2} = \dot{x}\ddot{x} \quad (1.2)$$

We proceed by integrating both sides of (1.2) over t .

By the fundamental theorem of calculus, the left hand side gives

$$\int_0^{t_0} \frac{d}{dt} \frac{\dot{x}^2}{2} dt = \frac{\dot{x}(t_0)^2}{2} - \frac{\dot{x}(0)^2}{2}$$

By substituting (1.1) for $\ddot{x}$ and using the change of variables formula, the right hand side gives

$$\begin{aligned} \int_0^{t_0} \dot{x}\ddot{x} dt &= -gr_0^2 \int_0^{t_0} \frac{\dot{x}}{(r_0 + x)^2} dt \\ &= -gr_0^2 \int_0^{x(t_0)} \frac{1}{(r_0 + x)^2} dx \\ &= gr_0^2 \left. \frac{1}{r_0 + x} \right|_0^{x(t_0)} \\ &= \frac{gr_0^2}{r_0 + x(t_0)} - gr_0 \end{aligned}$$

Combining both sides and solving for $\dot{x}(0)$ gives

$$\begin{aligned}\dot{x}(0) &= \sqrt{2 \left(gr_0 - \frac{gr_0^2}{r_0 + x(t_0)} + \frac{\dot{x}(t_0)^2}{2} \right)} \\ &\geq \sqrt{2gr_0 - \frac{2gr_0^2}{r_0 + x(t_0)}} \\ &= \sqrt{2gr_0 - \epsilon}\end{aligned}$$

where $\epsilon = \frac{2gr_0^2}{r_0 + x(t_0)}$. Since we have assumed that x increases unboundedly, escaping Earth, we can make ϵ arbitrarily small by increasing t_0 . Since the above holds for all ϵ , we have that the initial velocity

$$\begin{aligned}\dot{x}(0) &\geq \sqrt{2gr_0} \\ &= \sqrt{2 \cdot .0098 \frac{\text{km}}{\text{s}^2} \cdot 6371 \text{km}} \\ &= \sqrt{124.87 \frac{\text{km}^2}{\text{s}^2}} \\ &= 11.17 \frac{\text{km}}{\text{s}}\end{aligned}$$

Another way to show this is that in Hamiltonian phase space, any solution curve must lie in the level set curve

$$\frac{y^2}{2} + \frac{-gr_0^2}{r_0 + x} = E$$

where $U(x) = \frac{-gr_0^2}{r_0 + x}$ is the potential and E is an energy level. Thus we have that

$$E = \frac{\dot{x}(t)^2}{2} + \frac{-gr_0^2}{r_0 + x(t)} \geq \frac{-gr_0^2}{r_0 + x(t)}$$

Since we have assumed we can make $x(t)$ arbitrarily large, we can make $\frac{-gr_0^2}{r_0 + x(t)}$ arbitrarily close to 0 so that $E \geq 0$.

Thus we have that

$$\begin{aligned}\dot{x}(0) &= \sqrt{2 \left(E + \frac{gr_0^2}{r_0 + x(0)} \right)} \\ &= \sqrt{2E + 2gr_0} \\ &\geq \sqrt{2gr_0}\end{aligned}$$

which also demonstrates the result.

PROBLEM p. 18-4: Show that the time it takes to go from x_1 to x_2 (in one direction) is equal to

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E - U(x))}}$$

SOLUTION: Given a curve $c(t) = (x(t), y(t))$ such that $\dot{x}(t) = y(t)$, we have that for $y(t) \neq 0$

$$\begin{aligned} t_2 - t_1 &= \int_{t_1}^{t_2} dt \\ &= \int_{t_1}^{t_2} \frac{y(t)}{y(t)} dt \\ &= \int_{t_1}^{t_2} \frac{\dot{x}(t)}{\sqrt{2(E - U(x(t)))}} dt \\ &= \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E - U(x))}} \end{aligned}$$

by the change of variables formula for integrals.

The only question is whether this integral extends to the limit where $y(t_0) = 0$ and the potential is at its highest value $U(x_0) = E$ for $x_0 = x(t_0)$. We will show we can extend the integral to these points when $U'(x_0) \neq 0$.

Thus assume that $U'(x_0) \neq 0$ and in particular that $U'(x_0) < 0$. Because U' is continuous, we can find a neighborhood V of x_0 within which $U'(x) < 0$ for all $x \in V$. Thus in that neighborhood $U(x)$ is monotonically decreasing. Choose a point $x_1 > x_0$ such that $x_1 \in V$ and form the sequence

$$y_0 = x_1 - x_0, y_n = \frac{y_0}{2^n}$$

Then we have the following facts:

$$\begin{aligned} y_{n+1} &= \frac{y_0}{2 \cdot 2^n} = \frac{y_n}{2} \\ y_n - y_{n+1} &= y_n - \frac{y_n}{2} = \frac{y_n}{2} = y_{n+1} \\ \frac{y_{n+1}}{y_n} &= \frac{y_n}{2y_n} = \frac{1}{2} \\ \lim_{n \rightarrow \infty} y_n &= 0 \end{aligned}$$

Now consider the improper integral

$$\lim_{x \rightarrow x_0} \int_x^{x_1} \frac{dx}{\sqrt{2(E - U(x))}}$$

For any x , choose an n so that $y_n < x - x_0$ in which case $[y_{i+1} + x_0, y_i + x_0]$ for $i = 0, \dots, n-1$ is a partition of $[x, x_1]$. Since U is monotonically decreasing in V , $\sqrt{2(E - U(x))}$ is monotonically increasing, so that $\frac{1}{\sqrt{2(E - U(x))}}$ is monotonically decreasing. Thus the supremum of the function $\frac{1}{\sqrt{2(E - U(x))}}$ in each interval $[y_{i+1} + x_0, y_i + x_0]$ occurs at $y_{i+1} + x_0$. By Taylor's theorem, write $U(x) = E + U'(x_0)(x - x_0) + r(x)$ where $\lim_{x \rightarrow x_0} \frac{r(x)}{x - x_0} = 0$. Equivalently, $\lim_{x \rightarrow 0} \frac{r(x + x_0)}{x} = 0$. Then we can bound the integral by the upper partial Riemann sum as follows:

$$\begin{aligned}
 \int_x^{x_1} \frac{dx}{\sqrt{2(E - U(x))}} &\leq \sum_{i=1}^n \frac{(y_i + x_0) - (y_{i+1} + x_0)}{\sqrt{2(E - U(y_{i+1} + x_0))}} \\
 &= \sum_{i=1}^n \frac{y_i - y_{i+1}}{\sqrt{2(E - E - U'(x_0)(y_{i+1} + x_0 - x_0) - r(y_{i+1} + x_0))}} \\
 &= \sum_{i=1}^n \frac{y_{i+1}}{\sqrt{-2U'(x_0)(y_{i+1}) - 2r(y_{i+1} + x_0)}} \\
 &= \sum_{i=1}^n \frac{y_{i+1}}{\sqrt{y_{i+1}} \cdot \sqrt{-2U'(x_0) - 2\frac{r(y_{i+1} + x_0)}{y_{i+1}}}} \\
 &= \sum_{i=1}^n \frac{\sqrt{y_{i+1}}}{\sqrt{-2U'(x_0) - 2\frac{r(y_{i+1} + x_0)}{y_{i+1}}}}
 \end{aligned}$$

so that

$$\lim_{x \rightarrow 0} \int_x^{x_1} \frac{dx}{\sqrt{2(E - U(x))}} \leq \sum_{i=1}^{\infty} \frac{\sqrt{y_{i+1}}}{\sqrt{-2U'(x_0) - 2\frac{r(y_{i+1} + x_0)}{y_{i+1}}}}$$

Applying the ratio test to the right hand side of the above inequality shows that

$$\begin{aligned}
 &\lim_{i \rightarrow \infty} \frac{\sqrt{y_{i+1}}}{\sqrt{-2U'(x_0) - 2\frac{r(y_{i+1} + x_0)}{y_{i+1}}}} \cdot \frac{\sqrt{-2U'(x_0) - 2\frac{r(y_i + x_0)}{y_i}}}{\sqrt{y_i}} \\
 &= \lim_{i \rightarrow \infty} \sqrt{\frac{y_{i+1}}{y_i}} \cdot \sqrt{\frac{-2U'(x_0) - 2\frac{r(y_i + x_0)}{y_i}}{-2U'(x_0) - 2\frac{r(y_{i+1} + x_0)}{y_{i+1}}}} \\
 &= \sqrt{\frac{1}{2}} \sqrt{\frac{-2U'(x_0) - 2 \cdot 0}{-2U'(x_0) - 2 \cdot 0}} \\
 &= \sqrt{\frac{1}{2}} < 1
 \end{aligned}$$

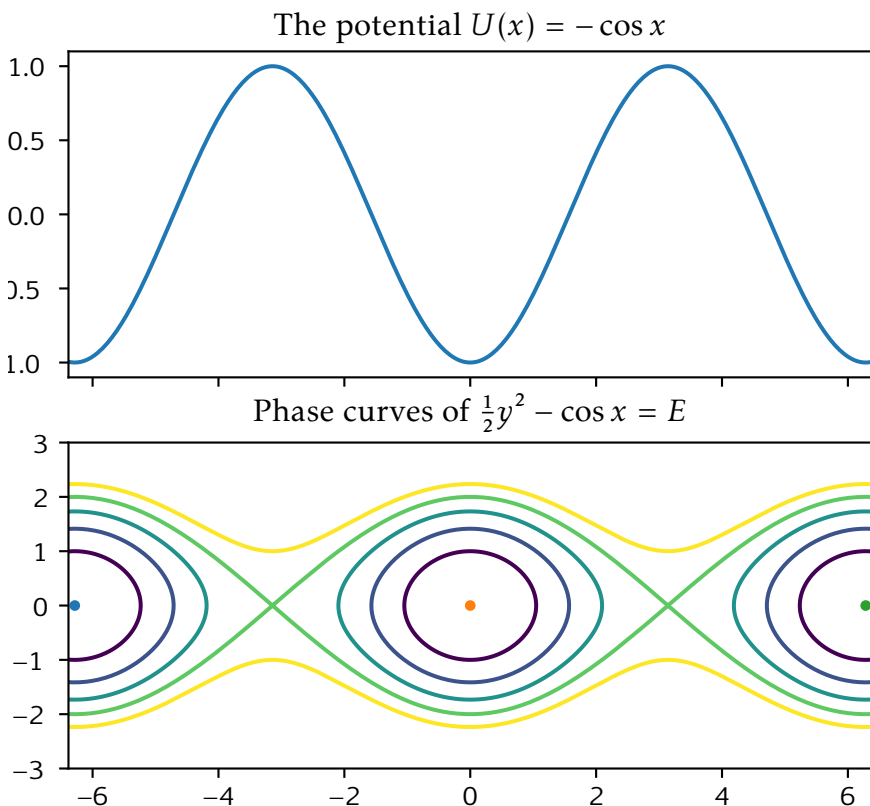


Figure 1.1: Problem 19-2: phase curves for $\ddot{x} = -\sin x$

so that the series converges. Thus the improper integral is bounded above and therefore also converges. Applying the same argument to the other side where $U'(x_1) > 0$ shows that the equation

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E - U(x))}}$$

applies even for $U(x_1) = U(x_2) = E$.

PROBLEM 19-2: Draw the phase curves for the “equation of an ideal planar pendulum”: $\ddot{x} = -\sin x$.

SOLUTION: The potential energy is $U(x) = -\cos x$ so that the phase curves are level sets of $\frac{1}{2}y^2 - \cos x = E$ for varying energy levels E . Various phase curves are shown in Figure 1.1.

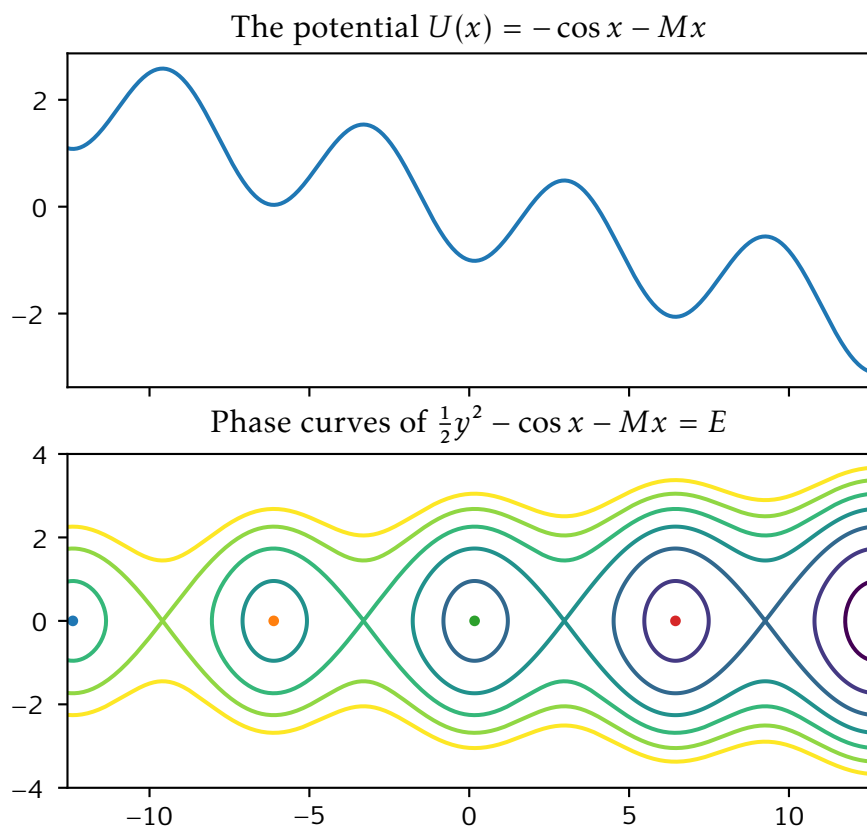


Figure 1.2: Problem 19-3: phase curves for $\ddot{x} = -\sin x + M$

PROBLEM 19-3: Draw the phase curves for the “equation of a pendulum on a rotating axis”: $\ddot{x} = -\sin x + M$.

SOLUTION: The potential energy is $U(x) = -\cos x - Mx$ so that the phase curves are level sets of $\frac{1}{2}y^2 - \cos x - Mx = E$ for varying energy levels E . Various phase curves are shown in Figure 1.2.

PROBLEM p. 19-4: Find the tangent lines to the branches of the critical level corresponding to the maximal potential energy $E = U(\xi)$.

This solution is entirely thanks to lolcats who explained it to me in the Libera chat ##math channel.

SOLUTION: Every phase curve lies in the level set

$$\frac{1}{2}y^2 + U(x) = E$$

A critical level is such that $U(\xi) = E$ is a local maximum at ξ . Thus we also have that $U'(\xi) = 0$ and that $U''(\xi)$ is negative in a neighborhood of ξ . Taylor's theorem with remainder allows us to write U as

$$\begin{aligned} U(x) &= U(\xi) + U'(\xi)(x - \xi) + \frac{1}{2}U''(\xi)(x - \xi)^2 + r(x) \\ &= E + \frac{1}{2}U''(\xi)(x - \xi)^2 + r(x) \end{aligned}$$

where the remainder term r satisfies $\lim_{x \rightarrow \xi} \frac{r(x)}{(x - \xi)^2} = 0$. Thus we have that

$$\begin{aligned} y(x)^2 &= 2(E - U(x)) \\ &= 2(E - E - \frac{1}{2}U''(\xi)(x - \xi)^2 - r(x)) \\ &= -U''(\xi)(x - \xi)^2 - 2r(x) \end{aligned}$$

so that

$$\frac{y(x)^2}{(x - \xi)^2} = -U''(\xi) - 2\frac{r(x)}{(x - \xi)^2}$$

and thus

$$\frac{y(x)}{x - \xi} = \pm \sqrt{-U''(\xi) - 2\frac{r(x)}{(x - \xi)^2}}$$

This is the slope of the secant line from $(\xi, 0)$ to $(x, y(x))$ so that taking the limit $x \rightarrow \xi$ gives us the slope of the tangent line at ξ :

$$\lim_{x \rightarrow \xi} \frac{y(x)}{x - \xi} = \pm \sqrt{-U''(\xi)}$$

Thus we have that

$$\pm \sqrt{-U''(\xi)}(x - \xi)$$

are the tangent lines to the level curve at ξ .

PROBLEM p. 21-1: Show that the system with potential energy $U = -x^4$ does not define a phase flow.

SOLUTION: Any potential phase flow g^t is composed of curves $c_p(t) = g^t p$ which satisfy the equations of motion:

$$\begin{aligned} \dot{x}_p(t) &= y_p(t) \\ \dot{y}_p(t) &= -\frac{dU}{dx}(x_p(t)) \\ &= 4(x_p(t))^3 \end{aligned}$$

where $c_p(t) = (x_p(t), y_p(t))$. We investigate one of these curves to show that it cannot give rise to a phase flow.

Consider the point $p = (1, \sqrt{2})$ which satisfies $\frac{1}{2}\sqrt{2}^2 - 1^4 = 1 - 1 = 0$ thus lying on the level curve corresponding to $E = 0$. We will derive a formula for $c_p(t) = (x_p(t), y_p(t))$. Since we have in this case that $U(x) = -x^4$, $E = 0$ and $x_p(0) = 1$, we use the result of Problem 18-4 as follows:

$$\begin{aligned}
 t &= t - 0 \\
 &= \int_{x_p(0)}^{x_p(t)} \frac{dx}{\sqrt{2(E - U(x))}} \\
 &= \int_1^{x_p(t)} \frac{dx}{\sqrt{2x^4}} \\
 &= \frac{1}{\sqrt{2}} \int_1^{x_p(t)} \frac{dx}{x^2} \\
 &= \frac{1}{\sqrt{2}} - \frac{1}{x} \Big|_1^{x_p(t)} \\
 &= \frac{1}{\sqrt{2}} \left(-\frac{1}{x_p(t)} + 1 \right)
 \end{aligned}$$

so that

$$x_p(t) = \frac{1}{1 - \sqrt{2}t}$$

Since $y = \sqrt{2(E - U(x))} = \sqrt{2x^4}$ we have that

$$\begin{aligned}
 y_p(t) &= \sqrt{2 \left(\frac{1}{1 - \sqrt{2}t} \right)^4} \\
 &= \frac{\sqrt{2}}{(1 - \sqrt{2}t)^2}
 \end{aligned}$$

from which we can verify that

$$\begin{aligned}
 \frac{d}{dt} x_p(t) &= -\frac{1}{(1 - \sqrt{2}t)^2} \cdot -\sqrt{2} \\
 &= \frac{\sqrt{2}}{(1 - \sqrt{2}t)^2} \\
 &= y_p(t)
 \end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}y_p(t) &= -\frac{2\sqrt{2}}{(1-\sqrt{2}t)^3} \cdot -\sqrt{2} \\ &= \frac{4}{(1-\sqrt{2}t)^3} \\ &= 4(x_p(t))^3\end{aligned}$$

so that

$$c_p(t) = \left(\frac{1}{1-\sqrt{2}t}, \frac{\sqrt{2}}{(1-\sqrt{2}t)^2} \right)$$

does indeed define the unique curve at $p = (1, \sqrt{2})$ that satisfies the equations of motion.

Since the denominators are 0 at $t = \frac{1}{\sqrt{2}}$, the maximal domain of definition of this integral curve is $]-\infty, \frac{1}{\sqrt{2}}[$ and we have that

$$\lim_{t \rightarrow \frac{1}{\sqrt{2}}} c_p(t) = (\infty, \infty)$$

Thus by fixing $t = \frac{1}{\sqrt{2}}$ and varying p we have that

$$\lim_{p \rightarrow (1, \sqrt{2})} g^{\frac{1}{\sqrt{2}}} p = (\infty, \infty)$$

by the theorem on smooth dependence of initial conditions, (Chapter 2, Corollary 4 of Arnold, *Ordinary Differential Equations*). But then $g^{1/\sqrt{2}}$ cannot be a diffeomorphism since there is no value we can assign at $p = (1, \sqrt{2})$ so that $g^{1/\sqrt{2}}$ will be continuous, much less smooth.

Figure 1.3 shows the vector field $y \frac{\partial}{\partial x} + 4x^3 \frac{\partial}{\partial y}$ within the square $[-1, 1] \times [-1, 1]$, various level sets of $\frac{1}{2}y^2 - x^4 = E$, the curve $c_p(t)$ in blue and the point $p = (1, \sqrt{2})$ at the red $\times$.

PROBLEM 24-3: Find the projection of the phase curves on the x_1, x_2 plane (i.e., draw the orbits of the motion of a point).

SOLUTION: The orbit of a point is an ellipse. I will show this in two ways, once via linear algebra and again via trigonometry. By problem 24-1, the trajectories are great circles of the sphere S^3 . The inclusion of the plane in $\mathbb{R}^4$ which maps the unit circle to the great circle is a linear transformation. And projecting down to two dimensions (x_1, x_2) is also linear so that the image of the trajectory is a linear transformation of the circle. Since the

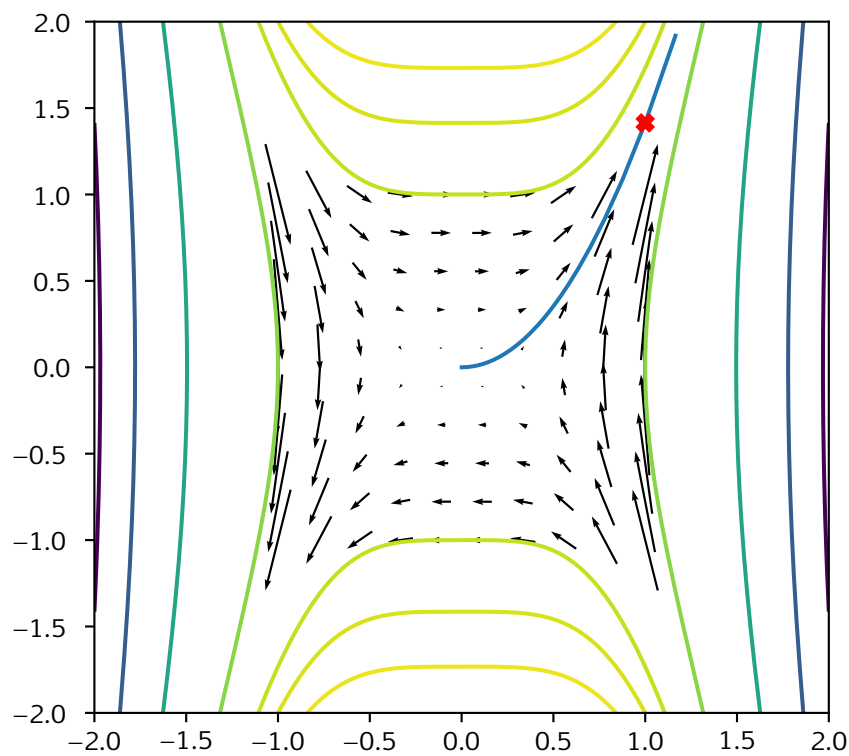


Figure 1.3: Problem 21-1

image of a circle under a linear transformation is an ellipse (see digression below), the trajectory is therefore an ellipse.

Now we show this again using trigonometry, using the following identities

$$a \cos t + b \sin t = c \cos(t - \theta)] \quad (1.3)$$

$$\cos(t - \theta) = \cos t \cos \theta + \sin t \sin \theta \quad (1.4)$$

In equation (1.3), we have that $c = \sqrt{a^2 + b^2}$ and θ is some angle (more or less $\arctan \frac{b}{a}$).

As the text mentions, the solution that passes through the point a_1, b_1, a_2, b_2 is

$$(x_1, y_1) = (a_1 \cos t + b_1 \sin t, -a_1 \sin t + b_1 \cos t)$$

$$(x_2, y_2) = (a_2 \cos t + b_2 \sin t, -a_2 \sin t + b_2 \cos t)$$

so that the position of the particle is given by

$$(x_1, x_2) = (a_1 \cos t + b_1 \sin t, a_2 \cos t + b_2 \sin t)$$

Using (1.3) we rewrite $x_i = a_i \cos t + b_i \sin t$ as $c_i \cos(t - \phi_i)$ for $i = 1, 2$.

Thus we have that $(x_1(t), x_2(t)) = (c_1 \cos(t - \phi_1), c_2 \cos(t - \phi_2))$. Then

$$\begin{aligned} \frac{x_2}{c_2} &= \cos(t - \phi_2) \\ &= \cos(t - \phi_2 + \phi_1 - \phi_1) \\ &= \cos(t - \phi_1 - \delta) \quad (\text{where } \delta = \phi_2 - \phi_1) \\ &= \cos(t - \phi_1) \cos \delta + \sin(t - \phi_1) \sin \delta \quad (\text{by identity (1.4)}) \\ &= \frac{x_1}{c_1} \cos \delta + \sqrt{1 - \cos^2(t - \phi_1)} \sin \delta \quad (\text{since } \cos^2 + \sin^2 = 1) \\ &= \frac{x_1}{c_1} \cos \delta + \sqrt{1 - \left(\frac{x_1}{c_1}\right)^2} \sin \delta \end{aligned}$$

Thus we have that

$$\frac{x_2}{c_2} - \frac{x_1}{c_1} \cos \delta = \sqrt{1 - \left(\frac{x_1}{c_1}\right)^2} \sin \delta$$

Squaring both sides gives

$$\begin{aligned} \left(\frac{x_2}{c_2}\right)^2 - 2 \frac{\cos \delta}{c_1 c_2} x_1 x_2 + \left(\frac{x_1}{c_1}\right)^2 \cos^2 \delta &= \left(1 - \left(\frac{x_1}{c_1}\right)^2\right) \sin^2 \delta \\ &= \sin^2 \delta - \left(\frac{x_1}{c_1}\right)^2 \sin^2 \delta \end{aligned}$$

so that

$$\begin{aligned}
 \sin^2 \delta &= \left(\frac{x_2}{c_2}\right)^2 - 2\frac{\cos \delta}{c_1 c_2} x_1 x_2 + \left(\frac{x_1}{c_1}\right)^2 \cos^2 \delta + \left(\frac{x_1}{c_1}\right)^2 \sin^2 \delta \\
 &= \left(\frac{x_2}{c_2}\right)^2 - 2\frac{\cos \delta}{c_1 c_2} x_1 x_2 + \left(\frac{x_1}{c_1}\right)^2 (\cos^2 \delta + \sin^2 \delta) \\
 &= \left(\frac{x_2}{c_2}\right)^2 - 2\frac{\cos \delta}{c_1 c_2} x_1 x_2 + \left(\frac{x_1}{c_1}\right)^2
 \end{aligned}$$

This is the equation of an ellipse as long as

$$\left(-2\frac{\cos \delta}{c_1 c_2}\right)^2 - 4\frac{1}{c_1^2 c_2^2} < 0 \quad (B^2 - 4AC < 0)$$

We have that

$$\begin{aligned}
 \left(-2\frac{\cos \delta}{c_1 c_2}\right)^2 - 4\frac{1}{c_1^2 c_2^2} &= \frac{4\cos^2 \delta}{(c_1 c_2)^2} - \frac{4}{(c_1 c_2)^2} \\
 &= \frac{4}{(c_1 c_2)^2} (\cos^2 \delta - 1) \\
 &\leq 0
 \end{aligned}$$

because $0 \leq \cos^2 \delta \leq 1$. Thus this is an ellipse as long as $\cos \delta \neq 1$, which is to say that δ cannot be 0 or π . In that case we would have that $\phi_1 = \phi_2$ or $\phi_1 + \pi = \phi_2$ and so the two cosines are synchronized along the same angle and the motion reduces to the one-dimensional case along a line.

A TRIGONOMETRIC DIGRESSION

The trigonometric identities could be taken at face value but demonstrating them geometrically is interesting so I'm including that here.

Both follow from the formula for rotating a point $(a, b) \in \mathbb{R}^2$ anti-clockwise given by

$$(a \cos t - b \sin t, a \sin t + b \cos t)$$

Essentially this formula follows from complex analysis, i.e., multiplying $(a, b) = a + bi$ by e^{it} . Then clockwise rotation by t , which is the formula we will actually use here, is

$$(a \cos t + b \sin t, -a \sin t + b \cos t) \tag{1.5}$$

which follows simply by substituting $-t$ into the previous formula.

To demonstrate identity (1.3), since (1.5) traces a circle of radius $c = \sqrt{a^2 + b^2}$, it must pass through the point $(c, 0)$. Thus, using (1.5) again, this path is also given by $(c \cos t, -c \sin t)$. Translating this path back by θ gives

$$(a \cos t + b \sin t, -a \sin t + b \cos t) = (c \cos(t - \theta), -c \sin(t - \theta))$$

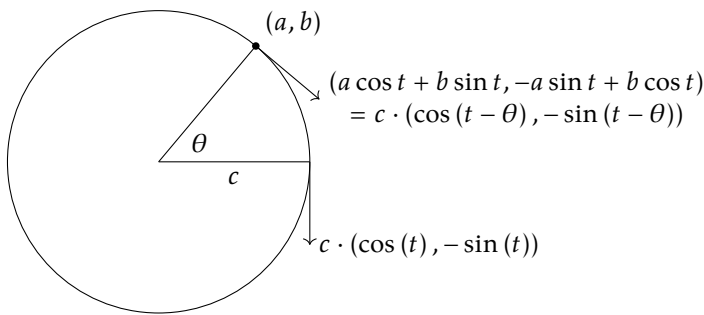


Figure 1.4: Proof of $a \cos t + b \sin t = c \cos(t - \theta)$

and the desired identity is the first component of this equality. This is depicted in Figure 1.4.

For the second identity (1.4) we clockwise rotate the point $(\cos(t), \sin(t))$ by θ which is both $(\cos(t - \theta), \sin(t - \theta))$ and also

$$(\cos t \cos \theta + \sin t \sin \theta, -\cos t \sin \theta + \sin t \cos \theta)$$

by (1.5). Thus the desired identity is the first component of the resultant equality.

A DIGRESSION ON ELLIPSES

It seems perfectly acceptable to take the image of a circle under a linear transformation as the definition of an ellipse. But I felt it was cheap to use two different definitions, whenever one or the other was most convenient.

So, I decided to define an ellipse to be the solution set of an equation

$$Ax^2 + Bxy + Cy^2 = D$$

for some constants A, B, C, D that satisfy $B^2 - 4AC < 0$ and then derive the fact that the image of a circle under a linear transformation satisfies such an equation. That derivation is included here.

Thus let $L = [a_{ij}]$ be a linear transformation and (x, y) a point on a circle, i.e., $x^2 + y^2 = 1$. Then the transformation of this point

$$A(x, y) = (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$$

ought to lie on an ellipse. For this to be the case, we ought to have constants

A, B, C such that

$$\begin{aligned}
 1 &= A(a_{11}x + a_{12}y)^2 + B(a_{11}x + a_{12}y)(a_{21}x + a_{22}y) + C(a_{21}x + a_{22}y)^2 \\
 &= A(a_{11}^2x^2 + 2a_{11}a_{12} + a_{12}^2y^2) + B(a_{11}a_{21}x^2 + (a_{11}a_{22} + a_{12}a_{21})xy) \\
 &\quad + C(a_{21}^2x^2 + 2a_{21}a_{22}xy + a_{22}^2y^2) \\
 &= (Aa_{11}^2 + Ba_{11}a_{21} + Ca_{21}^2)x^2 \\
 &\quad + (2Aa_{11}a_{12} + B(a_{11}a_{22} + a_{12}a_{21}) + 2Ca_{21}a_{22})xy \\
 &\quad + (Aa_{12}^2 + Ba_{12}a_{22} + Ca_{22}^2)y^2
 \end{aligned}$$

Thus if we can find A, B, C such that

$$\begin{aligned}
 Aa_{11}^2 + Ba_{11}a_{21} + Ca_{21}^2 &= 1 \\
 2Aa_{11}a_{12} + B(a_{11}a_{22} + a_{12}a_{21}) + 2Ca_{21}a_{22} &= 0 \\
 Aa_{12}^2 + Ba_{12}a_{22} + Ca_{22}^2 &= 1
 \end{aligned}$$

then the equality will hold since we are assuming that $x^2 + y^2 = 1$. But this is a system of linear equations

$$\begin{bmatrix} a_{11}^2 & a_{11}a_{21} & a_{21}^2 \\ 2a_{11}a_{12} & a_{11}a_{22} + a_{12}a_{21} & 2a_{21}a_{22} \\ a_{12}^2 & a_{12}a_{22} & a_{22}^2 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

with inverse

$$\frac{1}{\det L^2} \begin{bmatrix} a_{22}^2 & -a_{21}a_{22} & a_{21}^2 \\ -2a_{12}a_{22} & a_{11}a_{22} + a_{12}a_{21} & -2a_{11}a_{21} \\ a_{12}^2 & -a_{11}a_{12} & a_{11}^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

so that

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} a_{21}^2 + a_{22}^2 \\ -2(a_{11}a_{21} + a_{12}a_{22}) \\ a_{11}^2 + a_{12}^2 \end{bmatrix} \cdot \frac{1}{\det L^2}$$

Writing the equation of the ellipse as $Ax^2 + Bxy + Cy^2 = \det L^2$ we can check that

$$\begin{aligned}
 B^2 - 4AC &= 4(a_{11}^2a_{21}^2 + 2a_{11}a_{21}a_{12}a_{22} + a_{12}^2a_{22}^2) \\
 &\quad - 4(a_{21}^2a_{11}^2 + a_{21}^2a_{12} + a_{22}^2a_{11}^2 + a_{22}^2a_{12}^2) \\
 &= -4((a_{11}a_{22})^2 - 2(a_{11}a_{22})(a_{12}a_{21}) + (a_{12}a_{21})^2) \\
 &= -4(a_{11}a_{22} - a_{12}a_{21})^2 \\
 &= -4 \det L^2
 \end{aligned}$$

so that this is an ellipse as long as the linear transformation L is an isomorphism.